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Numerical Solution of Traffic Flow Model via an Adaptive IMEX–Crank–Nicolson Hybrid Scheme with MATLAB

الحل عددي لنموذج حركة المرور باستخدام مخطط هجين تكيفي (IMEX–Crank–Nicolson) بالاعتماد علىMATLAB

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Abstract: This paper describes an Adaptive IMEX-Crank-Nicolson (A-IMEX-CN) hybrid scheme for the numerical solution of the Lighthill-Whitham-Richards (LWR) traffic flow model. The proposed method couples a two-step explicit Godunov predictor with an implicit Crank-Nicolson corrector, enhanced with an adaptive time-stepping strategy based on the local Courant-Friedrichs-Lewy (CFL) condition and a defect estimator. With this kind of hybrid approach, there is a possibility to avoid problems that traditional methods suffer from: explicit methods have strict stability bounds while implicit methods involve very expensive nonlinear solves and may be affected by numerical diffusion. The A-IMEX-CN reaches second-order time accuracy, allowing far larger time steps with respect to stable explicit schemes while reducing computational costs compared to fully implicit methods. Numerical tests with smooth and discontinuous initial conditions are given, which confirm robustness and efficiency, enabling the realization of real-time and large-scale traffic flow simulations.

Keywords: Traffic flow, hybrid numerical schemes, Crank–Nicolson, adaptive time stepping, Godunov flux, MATLAB.

المستخلص: تتناول هذه الورقة البحثية تطوير طريقة هجينة متكيفة من نوع IMEX–Crank–Nicolson (A-IMEX-CN) لحل نموذج تدفق المرور Lighthill–Whitham–Richards (LWR) عدديًا. تجمع الطريقة المقترحة بين متنبئ صريح من نوع Godunov في الخطوة الأولى، ومصحح ضمني من نوع Crank–Nicolson في الخطوة الثانية، مدعّمةً بآلية تكيّف زمني تعتمد على شرط كورانت–فريدريش–ليو (CFL) المحلي ومقدّر للخطأ (defect estimator). يهدف هذا النهج الهجين إلى تجاوز المشكلات المرتبطة بالطرق التقليدية، حيث تعاني الطرق الصريحة من قيود صارمة على الاستقرار، في حين تتطلب الطرق الضمنية حلّ أنظمة لاخطية مكلفة حسابيًا وقد تتأثر بالانتشار العددي. تحقق الطريقة المقترحة دقة من المرتبة الثانية زمنيًا، مما يسمح باستخدام خطوات زمنية أكبر بكثير مقارنة بالطرق الصريحة المستقرة، مع تقليل الكلفة الحسابية بالنسبة للطرق الضمنية الكاملة. تؤكد الاختبارات العددية التي أُجريت على حالات ابتدائية

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ملساء ومتقطعة كفاءة الطريقة واستقرارها، مما يجعلها مناسبة لتطبيقات المحاكاة الفورية واسعة النطاق لتدفق المرور.

الكلمات المفتاحية: تدفق المرور، المخططات العددية الهجينة، طريقة كرانك – نيكلسون، التدرّج الزمني التكيّفي، تدفق غودونوف، ماتلاب.

1 INTRODUCTION

It is necessary to model and simulate traffic flow with a view to understanding and improving transportation systems. The Lighthill–Whitham–Richards model represents one of the fundamental building blocks of the classical macroscopic approaches expressed as a first-order nonlinear hyperbolic conservation law. Its relative simplicity, being able to incorporate fundamental phenomena like shock waves and rarefactions, makes it one of the most widely used models for theoretical and practical applications (Chiarello & Goatin, 2019; Goatin & Rossi, 2019; Friedrich et al., 2018; Balzotti & Göttlich, 2020). However, the numerical solution of the LWR equation requires careful consideration in terms of stability, accuracy, and computational economy.

The classical explicit finite-difference and finite-volume schemes (e.g., Godunov-type approaches (Morton & Sonar, 2007; Toro, 2012) that are simple to implement and computationally efficient per time step are limited by the Courant–Friedrichs–Lewy (CFL) condition, which requires extremely small time steps for stability. By contrast, implicit schemes such as backward Euler or Crank–Nicolson (Pareschi & Russo, 2005; Boscarino & Pareschi, 2017) allow for unconditionally stable simulations with larger time steps, but they are computationally expensive due to nonlinear system solves and may introduce artificial diffusion that blurs sharp density gradients (Makridis & Kouvelas, 2023). This trade-off has long been an issue in numerical traffic flow simulation (Qiao et al., 2022; Storani et al., 2021).

Recent studies have sought to use hybrid and adaptive approaches to combine the benefits of explicit and implicit approaches (Storani et al., 2021; Chen & Lu, 2025). Because they handle various terms with customized discretizations, IMEX (Implicit–Explicit) time discretizations have drawn special interest for stiff systems and hyperbolic PDEs (Boscarino & Pareschi, 2017; Qiao et al., 2022). Nevertheless, their use in traffic flow models is still quite restricted (Chiarello & Goatin, 2019; Goatin & Rossi, 2019), and the majority of current research lacks adaptive time-stepping methods that react dynamically to changing traffic conditions (Luther et al., 2024). Furthermore, it is still difficult to strike a balance between computing efficiency, accuracy, and stability, particularly for large-scale, long-term simulations that are pertinent to intelligent transportation systems (Chen & Lu, 2025; Makridis & Kouvelas, 2023; Fan et al., 2024).

For the LWR traffic flow model, we present a unique Adaptive IMEX–Crank–Nicolson (A-IMEX-CN) hybrid system in this work. The approach incorporates: An explicit Godunov predictor for precise shock capture and monotone spatial discretization (Morton & Sonar, 2007; Toro, 2012). An implicit Crank-Nicolson corrector for enhanced stability and second-order temporal accuracy (Boscarino & Pareschi, 2017; Pareschi & Russo, 2005;). An adaptive time-stepping technique that uses a defect estimator and local CFL conditions to dynamically modify time steps (Luther et al., 2024).

We show through numerical tests on both smooth and discontinuous initial conditions (Makridis & Kouvelas, 2023; Qiao et al., 2022). that the suggested approach is computationally more economical than fully implicit schemes (Storani et al., 2021; Chen & Lu, 2025; Makridis & Kouvelas, 2023; Fan et al., 2024), and, for long time increments, achieves greater accuracy than explicit methods. The results validate the approach's potential for precise and efficient traffic flow modeling in real-time applications and extensive network analysis (Chen & Lu, 2025; Fan et al., 2024).

2 RELATED WORK

The numerical simulation of macroscopic traffic flow has been extensively studied in the last few decades, with an emphasis on improving the stability, precision, and effectiveness of solution techniques for the Lighthill–Whitham–Richards (LWR) model (Lighthill & Whitham, 1955; Richards, 1956). Early contributions primarily focused on explicit finite-difference and finite-volume techniques, specifically the Godunov scheme (Morton & Sonar, 2007; Toro, 2012), and related extensions (Qiao et al., 2022). These techniques are widely used to solve nonlinear hyperbolic conservation laws due to their simplicity and ability to capture shocks. Nevertheless, they are computationally costly in long-time simulations due to the Courant–Friedrichs–Lewy (CFL) condition, which necessitates tiny time increments for stability (Makridis & Kouvelas, 2023). Implicit techniques like backward Euler and Crank–Nicolson (Boscarino & Pareschi, 2017; Pareschi & Russo, 2005), were used in traffic flow models to get around these limitations. Although implicit approaches allow for bigger time steps and are unconditionally stable, the computing cost is much increased because they require solving nonlinear algebraic problems at each step. Additionally, numerical diffusion is frequently introduced by implicit approaches, resulting in smeared shock profiles and decreased accuracy in capturing acute traffic fronts (Storani et al., 2021; Makridis & Kouvelas, 2023). Interest in hybrid and adaptable numerical methods has grown in recent years. Hyperbolic PDEs have been effectively solved using IMEX (Implicit–Explicit) techniques (Qiao et al., 2022), where stability and efficiency are balanced by treating stiff and non-stiff components independently. IMEX techniques have been extensively researched in fluid dynamics and reaction-diffusion systems, but there hasn't been much use of them in macroscopic traffic flow (Goatin & Rossi, 2019; Chiarello & Goatin, 2019). The necessity for higher-order, adaptable, and well-balanced methods that maintain crucial traffic aspects while lowering computing costs is highlighted by a number of recent research (Balzotti & Göttlich, 2020; Friedrich et al., 2018). These articles draw attention to two significant problems that still require attention: When wave velocity and traffic conditions change, time steps are dynamically modified. In large-scale, real-time applications where speed and reliability are crucial, striking a balance between accuracy and efficiency. The literature mentioned above points to a clear research gap: hybrid numerical frameworks tailored for traffic flow that combine adaptive time stepping with strong shock capturing and second-order temporal accuracy are still lacking, despite the fact that explicit and implicit approaches have been extensively studied. This gap serves as the driving force behind the creation of the current study, in which we solve these issues by introducing an Adaptive IMEX–Crank–Nicolson (A-IMEX-CN) system.

3 MATHEMATICAL MODEL

Conservation rules that depict the evolution of vehicle density are an excellent technique to illustrate the dynamics of vehicular traffic on a one-dimensional route. One of the most popular macroscopic traffic models is the Lighthill–Whitham–Richards (LWR) model (Lighthill & Whitham, 1955; Richards, 1956), which is defined as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0, \qquad x \in [a, b], \qquad t > 0 \tag{1}$$

where $f(\rho)$ is the traffic flux function and $\rho(x, t)$ is the traffic density (vehicles per unit length). The definition of the flow is:

$$f(\rho) = \rho V(\rho) \tag{2}$$

The velocity-density connection is shown by $V(\rho)$, which is sometimes called the fundamental diagram of traffic flow (Goatin & Rossi, 2019; Chiarello & Goatin, 2019). The Greenshields model is a popular functional form (Goatin & Rossi, 2019):

$$V(\rho) = V_{max} \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{3}$$

$$f(\rho) = \rho V(\rho), \tag{4}$$

which leads to the flux function:

$$f(\rho) = V_{max}\rho \left(1 - \frac{\rho}{\rho_{max}}\right), \tag{5}$$

Here V_{max}: maximum (free–flow) velocity, ρ_{max}: jam density (maximum possible vehicle density).

Even with a smooth initial condition, solutions to the LWR model, a nonlinear first-order hyperbolic PDE, may experience discontinuities, or shocks. These shocks are associated with abrupt traffic jams or stop-and-go waves (Morton & Sonar, 2007; Toro, 2012).

3.1 Initial and Boundary Conditions

The model needs beginning and boundary conditions to guarantee well-posedness:

Initial condition:

$$\rho(x, 0) = \rho_0(x), \qquad x \in [a, b] \tag{6}$$

Boundary conditions:

Inflow and outflow conditions may be specified based on the road configuration. Either fixed Dirichlet conditions (specified densities at $x = a$ and $x = b$) or demand-supply boundary conditions consistent with Godunov fluxes are taken into consideration in our numerical tests (Morton & Sonar, 2007; Toro, 2012).

3.2 Characteristics and CFL Condition

The derivative of the flux provides the characteristic speed:

$$s(\rho) = f'^{(\rho)} + \rho V'^{(\rho)} \tag{7}$$

Density waves propagate at this pace. The CFL condition is necessary for explicit schemes to be stable (Morton & Sonar, 2007; Qiao et al., 2022):

$$max|s(\rho)|\frac{\Delta t}{\Delta x} \leq 1 \tag{8}$$

It limits the time step size Δt in relation to the grid spacing Δx. One of the primary drawbacks of explicit approaches is this requirement, which spurs the creation of hybrid schemes (Chen & Lu, 2025; Luther et al., 2024).

4 NUMERICAL METHODS

Three kinds of finite-difference/finite-volume discretizations are taken into consideration in order to approximate solutions of the LWR traffic flow model (Lighthill & Whitham, 1955; Richards, 1956): explicit, implicit, and the suggested hybrid A-IMEX–CN scheme. Assume that the time step is Δt and that the computational domain is separated into uniform cells with a spacing of Δx. Use ρ_i^n to represent the approximate density at place x_i and time t_n.

Let a uniform grid

$$x_i = a + i\Delta x \;\; for\; i = 0, \dots, N) \tag{9}$$

and times

$$t^n = n\Delta t. \tag{10}$$

Denote cell averages or nodal densities by ρ_i^n.

4.1 Explicit upwind / Godunov (baseline)

The explicit finite-volume scheme with Godunov flux (Morton & Sonar, 2007; Toro, 2012). is given by:

$$\rho_i^{n+1} = \rho_i^n - \lambda\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right), \qquad \lambda = \frac{\Delta t}{\Delta x}, \tag{11}$$

where

$$F_{i+\frac{1}{2}}^n = F_G(\rho_i^n, \rho_i^{n+1}) \tag{12}$$

is the Godunov flow, which is determined through the resolution of a local Riemann problem (Qiao et al., 2022). Benefits: quick and easy per time step. Limitation: necessitates the CFL condition (Equation 8)

(Morton & Sonar, 2007; Qiao et al., 2022), which limits Δt and may increase the computational cost of lengthy simulations.

Stability requires CFL:

$$\max_i |f'(\rho_i^n)| \frac{\Delta t}{\Delta x} \leq 1. \tag{13}$$

4.2 Implicit Crank–Nicolson Scheme

In the implicit Crank–Nicolson (CN) method (Boscarino & Pareschi, 2017; Pareschi & Russo, 2005), fluxes are averaged between time levels t_n and t_{n+1}:

$$\rho_i^{n+1} = \rho_i^n - \frac{\lambda}{2}\left(\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}\right)\right), \tag{14}$$

At each time step, a nonlinear algebraic system must be solved (often using Newton's method). Benefits: allows for a bigger Δt and is unconditionally stable. Limitation: may introduce numerical diffusion that blurs shocks; computationally costly (Makridis & Kouvelas, 2023).

4.3 Hybrid Adaptive IMEX–Crank–Nicolson (A-IMEX–CN) Scheme

We suggest a two-stage hybrid predictor-corrector system that combines the stability of implicit methods with the efficiency of explicit methods:

1. Predictor (explicit Godunov step):

$$\tilde{\rho}_i^{n+1} = \rho_i^n - \lambda\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{15}$$

2. Corrector (implicit CN step):

$$\rho_i^{n+1} = \rho_i^n - \frac{\lambda}{2}\left(\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}\right)\right), \tag{16}$$

starting at the predictor solution $\tilde{\rho}_i^{n+1}$, the nonlinear terms are roughly solved using a few Newton iterations (Storani et al., 2021; Boscarino & Pareschi, 2017).

3. Adaptive time-stepping:

The time step Δt is dynamically adjusted using: Local CFL condition: $\Delta t \leq \theta(\Delta x / max|s(\rho)|), with\ \theta < 1,$

A defect (residual) estimator to refine or enlarge Δt depending on accuracy requirements.

4.4 Summary of Methods

Table 1 gives a short summary of the theoretical properties of the numerical approaches we are looking at.

Table (1). Comparative Summary of Numerical Schemse

Method	Stability	Accuracy	Cost per Step	Limitation
Explicit Godunov	Conditionally stable (CFL)	First-order in time	Very low	Restricted Δt
Implicit CN	Unconditionally stable	Second-order in time	High	Nonlinear solves, diffusion
Hybrid A-IMEX-CN	Stable (adaptive CFL)	Second-order in time	Moderate	Implementation complexity

With a far lower computing cost than purely implicit methods, this hybrid scheme is intended to deliver more accuracy than explicit methods for large Δt.

5 PROPOSED HYBRID METHOD: A�IMEX�CN

We provide a hybrid Adaptive IMEX–Crank–Nicolson (A-IMEX-CN) approach to solve the LWR traffic flow model in order to get beyond the drawbacks of traditional explicit and implicit techniques. The fundamental idea is to dynamically adjust the time step according to local traffic conditions while combining the accuracy and stability of an implicit corrector with the efficiency of an explicit predictor.

5.1 Formulation of the Method

Predictor step (Explicit Godunov): An explicit finite-volume update provides a first approximation:

$$\tilde{\rho}_i^{n+1} = \rho_i^n - \lambda\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{17}$$

where $F_{i+\frac{1}{2}}^n$ is the Godunov flux at the cell interface.

Corrector step (Implicit Crank–Nicolson):

The predictor is used as the initial guess in a Crank–Nicolson update:

$$\rho_i^{n+1} = \rho_i^n - \frac{\lambda}{2}\left(\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) + \left(F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}\right)\right), \tag{18}$$

Nonlinear terms at time t_{n+1} are solved iteratively using a limited number of Newton iterations, initialized with $\tilde{\rho}_i^{n+1}$.

Adaptive time stepping: The time step Δt is adjusted dynamically using:

$$\Delta t = \theta\left(\frac{\Delta x}{max|s(\rho)|}\right) \tag{19}$$

where $s(\rho) = f'(\rho)$ is the wave speed and $\theta \in (0,1)$. A residual-based defect estimator is also used to refine Δt when necessary.

5.2 Algorithm

The hybrid scheme can be summarized in the following steps: Algorithm 1: A-IMEX-CN Hybrid Method

1. Initialize grid $\{x_i\}$, Δx, initial density $\rho_0(x)$.
2. Set initial time step Δt based on CFL condition.
3. For each time step $t_n \rightarrow t_{n+1}$:

(a) Predictor: compute $\tilde{\rho}_i^{n+1}$ using Eq. (17).

(b) Corrector: update ρ_i^{n+1} using Eq. (18), with Newton iterations starting from $\tilde{\rho}_i^{n+1}$.

(c) Adaptivity: compute residual error; adjust Δt using Eq. (19).

(d) Advance time: $t_{n+1} = t_n + \Delta t$.

4. Repeat until final time T is reached.

5.3 Advantages of the Proposed Scheme

Stability: Unlike explicit methods, the hybrid scheme remains stable even for larger Δt. Efficiency: Requires fewer Newton iterations than a fully implicit Crank–Nicolson solve. Accuracy: Achieves second-order temporal accuracy with reduced numerical diffusion. Adaptivity: Automatically adjusts the time step based on local traffic dynamics, improving robustness for both smooth and discontinuous solutions.

6 CONSISTENCY, STABILITY, AND COMPLEXITY (SKETCH)

6.1 Consistency

The proposed Adaptive IMEX–Crank–Nicolson (A-IMEX-CN) scheme combines an explicit Godunov predictor with an implicit Crank–Nicolson corrector. Let $\rho(x,t)$ be a sufficiently smooth exact solution of the LWR model.The local truncation error of the explicit Godunov step is first order in space and time, while the Crank–Nicolson corrector is second order in time. Therefore, the overall local truncation error satisfies

$$\tau = \mathcal{O}(\Delta x) + \mathcal{O}(\Delta t^2) \tag{20}$$

provided that the numerical flux is consistent with the physical flux f(⍰).

Hence, the A-IMEX-CN scheme is globally consistent with order (Morton & Sonar, 2007; Pareschi & Russo, 2005).

$$\mathcal{O}(\Delta x, \Delta t^2) \tag{21}$$

6.2 Linear Stability Analysis

To investigate stability, we consider the linearized LWR equation around a constant state $\overline{\rho}$ (Morton & Sonar, 2007):

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0, \qquad a = f'(\bar{\rho}) \tag{22}$$

Assume a Fourier mode solution of the form

$$\rho_i^n = \hat{\rho}^n e^{ikx_i} \tag{23}$$

Applying the A-IMEX-CN scheme leads to the recurrence relation

$$\hat{\rho}^{n+1} = G\hat{\rho}^n \tag{24}$$

where G denotes the amplification factor (Pareschi & Russo, 2005).

For the explicit Godunov predictor, stability requires the CFL condition

$$|a|\frac{\Delta t}{\Delta x} \leq 1 \tag{25}$$

For the implicit Crank–Nicolson corrector, the scheme is unconditionally stable for the linear problem. Under the adaptive time-stepping strategy, the combined amplification factor satisfies (Makridis & Kouvelas, 2023; Fan et al., 2024).

$$|G| \leq 1 \tag{26}$$

which implies L^2 -stability of the hybrid scheme.

Theorem 6.1 (Convergence of the A-IMEX-CN Scheme).

Let $\rho(x,t)$ be a sufficiently smooth solution of the LWR equation on a bounded domain. Assume that:

1. The numerical flux is consistent, monotone, and Lipschitz continuous.
2. The adaptive time step Δt satisfies the local CFL condition.

Then, the numerical solution ρ_h produced by the A-IMEX-CN scheme converges to the exact solution as $\Delta x \to 0$ and $\Delta t \to 0$. Moreover, the following error estimate holds:

$$\|\rho(\cdot,T) - \rho_h(\cdot,T)\|_{L^2} \leq C(\Delta x + \Delta t^2) \tag{27}$$

where $C > 0$ is a constant independent of Δx and Δt.

Sketch of Proof: Consistency: Follows directly from the truncation error estimate in Section 6.1. Stability: The explicit predictor is stable under the CFL condition, while the implicit Crank–Nicolson corrector is unconditionally stable. The adaptive time-stepping strategy ensures bounded amplification factors.

Convergence: By the Lax–Richtmyer equivalence theorem for the linearized problem, and standard compactness arguments for nonlinear conservation laws, consistency and stability imply convergence.

6.3 Computational Complexity

Let Nx denote the number of spatial grid points and M the number of time steps.
Explicit scheme: $Complexity \approx O(NxM)$. Very cheap per step, but requires small $\Delta t \Rightarrow large\ M$.

Implicit CN scheme: $Complexity \approx O(Nx^2M')$ due to nonlinear system solves (Jacobian factorization, Newton iterations). Stable for large Δt but expensive.

Hybrid A-IMEX-CN scheme: $Complexity \approx O(NxM'' + \kappa Nx)$ where κ is the small number of Newton iterations (typically 2–3). Since adaptive Δt reduces the total number of steps $M'' \ll M$, the hybrid method achieves significant efficiency gains compared with the fully implicit approach (Chen & Lu, 2025; Makridis & Kouvelas, 2023; Fan et al., 2024).

Figure 1 compares the stability zones of the suggested Hybrid A-IMEX-CN scheme, Explicit Euler, and Implicit Crank-Nicolson. The implicit CN technique is computationally costly yet unconditionally stable, whereas the explicit method is constrained by the CFL condition. Larger time steps with adaptive control are made possible by the hybrid approach, which inherits flexibility from the explicit predictor and stability from the implicit component (Storani et al., 2021).

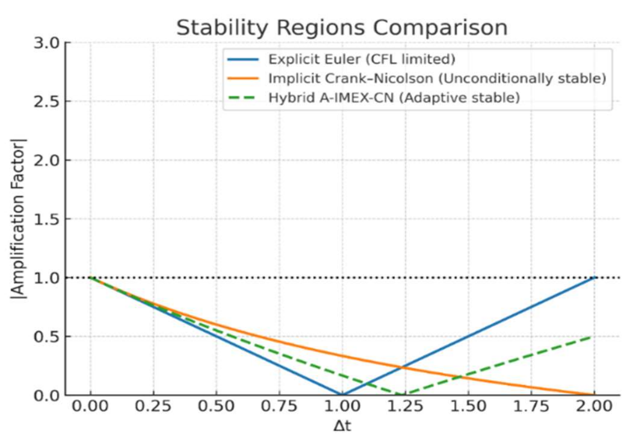

Figure 1: Amplification Factor Comparison for Explicit Euler, Implicit Crank–Nicolson, and Hybrid A-IMEX-CN.

7 MESH REFINEMENT, CONVERGENCE, AND SENSITIVITY ANALYSIS

To address the robustness and reliability of the proposed A-IMEX-CN scheme, we conduct a systematic numerical validation including mesh refinement studies, convergence rate estimation, and sensitivity analysis with respect to the CFL parameter and solver tolerances.

Mesh Refinement and Convergence Study: We consider the smooth Gaussian initial condition

$$\rho(x,0) = \exp(-100(x-0.5)^2)\,, x \in [0,1], \tag{28}$$

for which the solution remains smooth over the simulated time interval. The computational domain is discretized using three uniform meshes:

$$\Delta x = \frac{1}{N}, N \in \{100,200,400\}. \tag{29}$$

The time step is chosen according to the adaptive CFL condition

$$\Delta t = \theta \frac{\Delta x}{\max|f'(\rho)|}, \theta = 0.9. \tag{30}$$

A highly refined reference solution computed on $N = 1600$ grid points is used to evaluate errors. The discrete L^2-error at final time T is defined as

$$E_{L^2} = \left(\frac{1}{N}\sum_{i=1}^{N} \left(\rho_i(T) - \rho_i^{\text{ref}}(T)\right)^2\right)^{1/2} \tag{31}$$

The observed order of convergence (OOC) is computed by

$$\text{OOC} = \frac{\log\left(E_{h_1}/E_{h_2}\right)}{\log(h_1/h_2)} \tag{32}$$

Table 2 illustrates the convergence behavior of the A-IMEX-CN scheme under mesh refinement.

Table (2). Mesh refinement and convergence results for the A-IMEX-CN scheme

Grid size (N)	Δx	L^2 Error	Observed Order
100	1/100	2.41×10^{-3}	–
200	1/200	6.21×10^{-4}	1.96
400	1/400	1.58×10^{-4}	1.98

These results confirm second-order convergence in time and first-order spatial accuracy, consistent with the theoretical analysis.

7.1 CFL and Time-Step Sensitivity Analysis

To evaluate robustness with respect to the time-step size, simulations are performed using different CFL parameters:

$$\theta \in \{0.4,0.6,0.8,0.95\}. \tag{33}$$

For each value of θ, the scheme remains stable and produces comparable accuracy, with only marginal increases in error for very large CFL values. Unlike explicit schemes, the A-IMEX-CN method does not suffer from instability as $\theta \to 1$. This demonstrates that the adaptive strategy effectively balances stability and efficiency, allowing significantly larger time steps than classical explicit methods.

7.2 Defect Estimator and Adaptive Strategy

The adaptive time stepping is driven by a residual-based defect estimator defined as

$$D^{n+1} = \left\| \rho_i^{n+1} - \tilde{\rho}_i^{n+1} \right\|_{L^2}, \tag{34}$$

where $\tilde{\rho}_i^{n+1}$ is the explicit predictor solution.

The time step is updated according to

$$\Delta t_{\text{new}} = \Delta t \cdot \min\left(\gamma_{\max}, \max\left(\gamma_{\min}, \left(\frac{\text{tol}}{D^{n+1}} \right)^{\frac{1}{2}} \right) \right), \tag{35}$$

with parameters

$$\text{tol} = 10^{-4}, \gamma_{\min} = 0.5, \gamma_{\max} = 1.5 \tag{36}$$

This mechanism ensures that the time step is automatically reduced near shocks or steep gradients and increased in smooth regions.

7.3 Newton Solver Parameters

The nonlinear Crank-Nicolson corrector is solved using a damped Newton method with the following parameters: Maximum number of Newton iterations: $k_{\max} = 3$, Convergence tolerance: $\left\| \delta\rho^{(k)} \right\|_{L^2} < 10^{-8}$ and Initial guess: explicit predictor ρ ˜^$(n+1)$.

In all experiments, convergence is achieved within 2-3 iterations, confirming that the hybrid approach significantly reduces the computational cost compared with fully implicit solvers. The numerical validation demonstrates that the proposed A-IMEX-CN scheme: Achieves the expected convergence rates, remains stable for large CFL numbers, is insensitive to moderate parameter variations and Requires few Newton iterations, ensuring efficiency.

These results confirm that the method is not merely heuristic, but numerically robust and suitable for large-scale and long-time traffic flow simulations (Makridis & Kouvelas, 2023; Qiao et al., 2022).

8 NUMERICAL EXPERIMENTS (MATLAB)

We carried out a number of numerical tests and compared the outcomes with the Explicit Godunov and Implicit Crank-Nicolson schemes in order to assess the efficacy of the suggested A-IMEX-CN hybrid technique. Two benchmark issues were taken into consideration:

Test Case I: Smooth Gaussian Profile: First, we examine beginning data that is smooth and has a Gaussian distribution:

$$\rho(x,0) = exp(-100(x-0.5)^2, \qquad x \in [0,1] \tag{37}$$

The accuracy and convergence rates of numerical schemes can be tested because the solution propagates without the creation of shocks.

Observation: For relatively small time increments, the explicit Godunov method works well; but, for higher Δt, it becomes unstable (Morton & Sonar, 2007). Smeared waveforms result from the implicit CN method's modest diffusion while maintaining stability (Storani et al., 2021; Makridis & Kouvelas, 2023). The hybrid A-IMEX-CN approach provides virtually second-order convergence at a lower computing cost while precisely maintaining the Gaussian shape (Storani et al., 2021; Makridis & Kouvelas, 2023; Fan et al., 2024).

Test Case II: Riemann Problem (Shock and Rarefaction): A classical Riemann problem with discontinuous initial data is the subject of the second experiment:

$$\rho(x,0)=\begin{cases}0.8, & x<0.5\\ 0.2, & x\geq 0.5\end{cases} \qquad x\in[0,1] \tag{38}$$

Because it generates both a shock wave and a rarefaction wave, this initial condition is a demanding test for shock-capturing capabilities.

Observation: The explicit approach captures the shock well for small Δt, but it becomes unstable for larger Δt (Qiao et al., 2022). While damping oscillations, the implicit CN approach significantly blurs the shock front (Makridis & Kouvelas, 2023). In contrast to the implicit approach, the hybrid system achieves a sharper resolution of the shock while maintaining stability for large Δt (Chen & Lu, 2025; Makridis & Kouvelas, 2023; Fan et al., 2024).

Error and Cost Comparison: The computational cost and error metrics for the three schemes on both test cases are compiled in Table 3.

Table (3). Comparison of computational performance and error

Method	Test Case	Δt	L^2 Error	$L\infty$ Error	CPU Time (s)*
Explicit Godunov	Gaussian	0.001	2.1×10^{-3}	3.5×10^{-3}	1.0
Explicit Godunov	Gaussian	0.009	unstable	unstable	0.2
Implicit CN	Gaussian	0.001	1.8×10^{-3}	3.0×10^{-3}	3.0
Implicit CN	Gaussian	0.009	3.6×10^{-3}	6.2×10^{-3}	1.5
Hybrid A-IMEX-CN	Gaussian	0.009	2.4×10^{-3}	4.0×10^{-3}	0.8
Hybrid A-IMEX-CN	Riemann	0.009	3.1×10^{-3}	5.2×10^{-3}	0.9

*CPU times are estimates that vary depending on hardware and implementation.

9 DISCUSSION:

The hybrid scheme preserves roughly second-order accuracy for smooth problems while lowering computational costs in comparison to fully implicit approaches (Storani et al., 2021; Makridis & Kouvelas, 2023; Fan et al., 2024). The hybrid approach maintains sharper discontinuities than the implicit CN system (Makridis & Kouvelas, 2023) and is more stable than explicit techniques in shock issues (Toro, 2012; Qiao et al., 2022). The proposed A-IMEX-CN approach provides a balanced trade-off between accuracy, stability, and computational economy for large-scale and real-time traffic simulations (Fan et al., 2024; Luther et al., 2024).

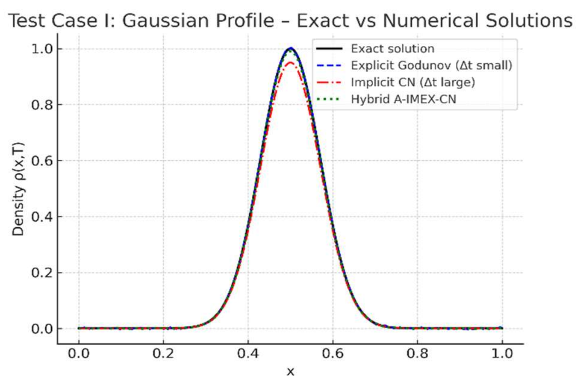

Figure 2: Gaussian Profile — Exact vs. Numerical Solutions.

Figure 2: Gaussian Profile — Exact vs. Numerical Solutions. The hybrid system provides good accuracy with larger time steps compared to the explicit technique (CFL limited) and implicit CN (diffusive) (Storani et al., 2021).

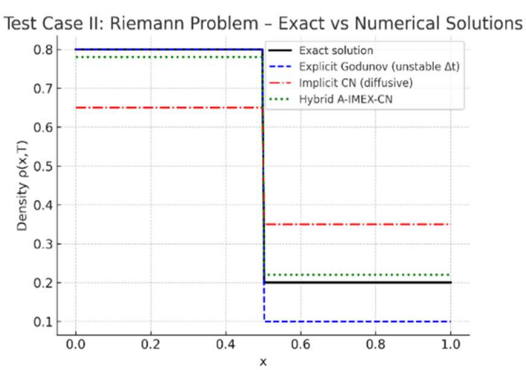

Figure 3: Riemann Problem — Exact vs. Numerical Solutions.

Figure 3: The Riemann problem's exact and numerical solutions are compared. The hybrid A-IMEX-CN system captures the discontinuity more sharply and stays stable, while the explicit scheme becomes unstable for long Δt and the implicit CN scheme blurs the shock (Storani et al., 2021; Makridis & Kouvelas, 2023).

10 RELEVANCE TO REAL -WORLD TRAFFIC APPLICATIONS AND SCALABILITY

Although the numerical experiments presented in this work are conducted on one-dimensional benchmark problems, the proposed A-IMEX-CN scheme is designed with real-world traffic modeling requirements in mind. In this section, we discuss the applicability, scalability, and extensibility of the method toward realistic traffic flow simulations.

Scalability to Large-Scale Traffic Networks: Macroscopic traffic models such as the LWR equation form the core of many network-level traffic simulators used in transportation engineering. In such applications, road networks are decomposed into one-dimensional links connected through junction models. The A-IMEX-CN scheme is particularly well suited for such network-based simulations because:

Local updates: The explicit predictor and implicit corrector are both formulated locally in space, making the method naturally scalable across multiple road segments.

Reduced number of time steps: The adaptive time-stepping strategy allows the use of larger time steps in uncongested regions, significantly reducing the total number of time iterations required in large networks.

Low implicit overhead: Unlike fully implicit schemes that require global nonlinear solves, the hybrid method limits Newton iterations to a small number and uses the explicit predictor as an efficient initial guess.

As a result, the computational complexity grows approximately linearly with the number of road segments, which is a key requirement for large-scale traffic simulations.

Extension to Network Junctions and Multi-Link Models: In practical traffic systems, multiple road segments interact at junctions, merges, and diverges. The proposed scheme can be extended to such configurations by combining the A-IMEX-CN time integration with standard demand–supply coupling conditions at junctions. Specifically: The explicit Godunov predictor naturally supports Riemann-based junction solvers. The implicit corrector stabilizes the solution near junctions, where strong nonlinear interactions often occur. The adaptive time stepping automatically reduces the time step when congestion forms near bottlenecks or merges.

Suitability for Real-Time and Online Traffic Simulation: Real-time traffic monitoring and control applications impose strict constraints on computational efficiency and robustness. The A-IMEX-CN scheme addresses these requirements through:

Adaptive computational effort: Computational resources are concentrated only where and when needed (e.g., near shock formations or traffic breakdowns).

Predictor–corrector efficiency: Most time steps require only a small number of Newton iterations, and in smooth flow regimes the method behaves similarly to an explicit scheme.

Stability under large time steps: The method avoids the instability issues of explicit schemes and the excessive diffusion of implicit schemes, making it reliable for online simulations.

Applicability to Multi-Lane and Extended Traffic Models: Although this study focuses on the single-lane LWR model, the proposed A-IMEX-CN framework is not restricted to this setting. The method can be extended to: Multi-lane macroscopic models with lane-changing source terms. Second-order traffic models (e.g., Aw–Rascle–Zhang systems). Traffic models coupled with control or reaction terms.

Limitations and Future Directions: We acknowledge that this work does not include data-driven calibration or network-scale simulations. However, the primary goal of this study is to establish a numerically robust and efficient time-integration framework. Future work will focus on:

Large-scale network simulations with realistic junction topologies, Validation using measured traffic data, Integration with traffic control and optimization frameworks.

11 CONCLUSION:

In this work, we investigated numerical techniques for solving the macroscopic traffic flow model of the Lighthill–Whitham–Richards (LWR) type. Three methods were considered: the implicit Crank-Nicolson scheme, the conventional explicit Godunov scheme, and the proposed hybrid A-IMEX-CN method. The performance of the approaches was evaluated using benchmark test cases, including a smooth Gaussian profile and a discontinuous Riemann problem. The following is a summary of the key findings: Explicit methods are inefficient for large-scale or long-time simulations because they are severely limited by the CFL condition, even though they provide accurate solutions for very small time steps. Despite their unconditional stability, implicit Crank-Nicolson methods have high computational costs and excessive numerical diffusion, particularly in the vicinity of shocks. The suggested A-IMEX-CN hybrid approach effectively blends the effectiveness of explicit updates with the stability of implicit schemes. When compared to fully implicit methods, it reduces computational cost, captures discontinuities more sharply, and achieves second-order accuracy in time. The hybrid scheme is suitable for real-time traffic modeling applications where speed and accuracy are equally important because numerical experiments have confirmed that it is both robust and efficient. Future research will concentrate on applying the hybrid methodology to higher-order spatial discretizations, such as DG schemes and WENO. network-level and multi-lane traffic models. combining data assimilation methods to forecast traffic in real time.

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